z.z. $\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-x^2-y^2} dx\right) dy = \pi$

Bew:

Es sei $\Omega = \mathbb{R}_{>0} \times (0, 2\pi)$ u. $\phi: \Omega \to \mathbb{R}^2$, ~~$(r,\varphi) \mapsto$~~ $r\cos r$

$r(\cos^2(\varphi) + \sin^2(\varphi)) = r$

$$\Rightarrow \det D\phi(r,\varphi) = \begin{vmatrix} \cos(\varphi) & -r\sin(\varphi) \\ \sin(\varphi) & r\cos(\varphi) \end{vmatrix} = r(\cos^2(\varphi) + \sin^2(\varphi)) = r$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dx\, dy = \int_{\phi(\Omega)} e^{-x^2-y^2} dx\, dy$$

$$\overset{\text{Transformationssatz}}{=} \int_{\Omega} e^{-(r\cos(\varphi))^2 - (r\cdot\sin(\varphi))^2} \cdot \det D\phi(r,\varphi)\, dr\, d\varphi$$

$$= \int_{\Omega} e^{-r^2} \cdot r\, dr\, d\varphi$$

$$= \int_0^{2\pi}\int_0^{\infty} r e^{-r^2} dr\, d\varphi$$

$$= \int_0^{2\pi} \frac{1}{2}\, d\varphi = \pi$$

□ q.e.d.