19.1, Aufgabe 16 von: Sebastian Neumann, Malte Pfalzgraf, Melanie Schäfer

Es seien $f, g \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$.

z.z.: $\Delta(f \cdot g) = f \cdot \Delta g + 2 \langle \nabla f, \nabla g \rangle + g \cdot \Delta f$

Beweis:

$f, g : \Omega = \mathbb{R}^n \longrightarrow \mathbb{R}$

$$\begin{aligned}\Delta(f \cdot g) &= \operatorname{div} \operatorname{grad}(f \cdot g) \\ &= \operatorname{div} \nabla(f \cdot g) \\ &\overset{(*)}{=} \operatorname{div}(f \cdot \nabla g + g \cdot \nabla f)\end{aligned}$$

(*) Produktregel für $\nabla(f \cdot g)$:

$$\nabla(f \cdot g) = \left(\frac{\partial(f \cdot g)}{\partial x_1}, \frac{\partial(f \cdot g)}{\partial x_2}, \ldots, \frac{\partial(f \cdot g)}{\partial x_n} \right)^T$$

komponentenweise Produktregel

$$= \left(f \cdot \frac{\partial g}{\partial x_1} + g \cdot \frac{\partial f}{\partial x_1}, \ldots, f \cdot \frac{\partial g}{\partial x_n} + g \cdot \frac{\partial f}{\partial x_n} \right)^T$$

$$= f \cdot \begin{pmatrix} \frac{\partial g}{\partial x_1} \\ \vdots \\ \frac{\partial g}{\partial x_n} \end{pmatrix} + g \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$$

$$= f \cdot \nabla g + g \cdot \nabla f$$

$$\overset{\substack{A.11 \\ (ii)}}{=} \operatorname{div}(f \cdot \nabla g) + \operatorname{div}(g \cdot \nabla f)$$

$$= \operatorname{div} \begin{pmatrix} f \cdot \frac{\partial g}{\partial x_1} \\ \vdots \\ f \cdot \frac{\partial g}{\partial x_n} \end{pmatrix} + \operatorname{div} \begin{pmatrix} g \cdot \frac{\partial f}{\partial x_1} \\ \vdots \\ g \cdot \frac{\partial f}{\partial x_n} \end{pmatrix}$$

Def. 19.1.2

$$= \left(\frac{\partial(f \cdot \partial g)}{\partial x_1^2} + \ldots + \frac{\partial(f \cdot \partial g)}{\partial x_n^2} \right) + \left(\frac{\partial(g \cdot \partial f)}{\partial x_1^2} + \ldots + \frac{\partial(g \cdot \partial f)}{\partial x_n^2} \right)$$

Produktregel in jedem Summanden

$$= \left(\frac{f \cdot \partial^2 g}{\partial x_1^2} + \frac{\partial f \, \partial g}{\partial x_1^2} \right) + \ldots + \left(\frac{f \, \partial^2 g}{\partial x_n^2} + \frac{\partial f \, \partial g}{\partial x_n^2} \right) + \left(\frac{g \, \partial^2 f}{\partial x_1^2} + \frac{\partial f \, \partial g}{\partial x_1^2} \right) + \ldots + \left(\frac{g \cdot \partial^2 f}{\partial x_n^2} + \frac{\partial f \, \partial g}{\partial x_n^2} \right)$$

$$= f \cdot \sum_{i=1}^{n} \frac{\partial^2 g}{\partial x_i^2} + \sum_{i=1}^{n} \frac{\partial f \, \partial g}{\partial x_i^2} + g \cdot \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} + \sum_{i=1}^{n} \frac{\partial f \, \partial g}{\partial x_i^2}$$

$$= f \cdot \operatorname{div} \operatorname{grad} g + 2 \sum_{i=1}^{n} \frac{\partial f \, \partial g}{\partial x_i^2} + g \cdot \operatorname{div} \operatorname{grad} f$$

$$= f \cdot \Delta g + 2 \langle \nabla f, \nabla g \rangle + g \cdot \Delta f \quad \square$$