Aufgabe 19.1.11 (Anne Frolich, Till Preuß, Catherin Beyer)

$f, g: \mathbb{R}^3 \to \mathbb{R}$ stetig diffbar $a, b: \mathbb{R}^3 \to \mathbb{R}^3$ stetig diffbare Vektorfelder, $\lambda, \mu \in \mathbb{R}$

zz: i) $D(\lambda f + \mu g) = \lambda Df + \mu Dg$

ii) $\operatorname{div}(\lambda a + \mu b) = \lambda \operatorname{div} a + \mu \operatorname{div} b$

iii) $\operatorname{rot}(\lambda a + \mu b) = \lambda \operatorname{rot} a + \mu \operatorname{rot} b$

Bew: i) $$D(\lambda f + \mu g) = \left( \frac{\partial(\lambda f + \mu g)}{\partial x}, \frac{\partial(\lambda f + \mu g)}{\partial y}, \frac{\partial(\lambda f + \mu g)}{\partial z} \right)$$

$$= \left( \frac{\lambda \partial f}{\partial x} + \frac{\mu \partial g}{\partial x}, \frac{\lambda \partial f}{\partial y} + \mu \frac{\partial g}{\partial y}, \frac{\lambda \partial f}{\partial z} + \mu \frac{\partial g}{\partial z} \right)$$

$$= \lambda Df + \mu Dg$$

ii) $$\operatorname{div}(\lambda a + \mu b) = \sum_{i=1}^{3} \frac{\partial(\lambda a_i + \mu b_i)}{\partial x_i} = \frac{\partial(\lambda a_1 + \mu b_1)}{\partial x_1} + \frac{\partial(\lambda a_2 + \mu b_2)}{\partial x_2} + \frac{\partial(\lambda a_3 + \mu b_3)}{\partial x_3}$$

$$= \frac{\lambda \partial a_1}{\partial x_1} + \mu \frac{\partial b_1}{\partial x_1} + \lambda \frac{\partial a_2}{\partial x_2} + \mu \frac{\partial b_2}{\partial x_2} + \lambda \frac{\partial a_3}{\partial x_3} + \mu \frac{\partial b_3}{\partial x_3}$$

$$= \lambda \operatorname{div} a + \mu \operatorname{div} b$$

iii) $$\operatorname{rot}(\lambda a + \mu b) = \left( \frac{\partial(\lambda a_3 + \mu b_3)}{\partial x_2} - \frac{\partial(\lambda a_2 + \mu b_2)}{\partial x_3}, \frac{\partial(\lambda a_1 + \mu b_1)}{\partial x_3} - \frac{\partial(\lambda a_3 + \mu b_3)}{\partial x_1}, \frac{\partial(\lambda a_2 + \mu b_2)}{\partial x_1} - \frac{\partial(\lambda a_1 + \mu b_1)}{\partial x_2} \right)$$

$$= \left( \lambda \left( \frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3} \right) + \mu \left( \frac{\partial b_3}{\partial x_2} - \frac{\partial b_2}{\partial x_3} \right), \lambda \left( \frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_2} \right) + \mu \left( \frac{\partial b_1}{\partial x_3} - \frac{\partial b_3}{\partial x_2} \right), \lambda \left( \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2} \right) + \mu \left( \frac{\partial b_2}{\partial x_1} - \frac{\partial b_1}{\partial x_2} \right) \right)$$

$$= \lambda \operatorname{rot} a + \mu \operatorname{rot} b$$