Aufgabe 8

Eisel, Weirich, Wimmer

i) $f: \mathbb{R}^2 \to \mathbb{R}^2$ vermöge $f(x,y) = (x^2+y^2, 2xy)$

Aus $\frac{\partial f_1}{\partial y} = 2y = \frac{\partial f_2}{\partial x}$ & sternförmig folgt

f ist Gradientenfeld.

Potential:

$$\varphi(x,y) = \int_0^1 \langle f((x_0,y_0) + t(x-x_0, y-y_0)), (x-x_0, y-y_0) \rangle \, dt$$

$$\overset{x_0,y_0=0}{=} \int_0^1 \langle f(t(x,y)), (x,y) \rangle \, dt$$

$$= \int_0^1 \langle ((tx)^2 + (ty)^2, (2t^2xy)), (x,y) \rangle \, dt$$

$$= \int_0^1 t^2x^3 + \underbrace{t^2y^2x + 2t^2xy^2}_{=3t^2y^2x} \, dt$$

$$= \frac{1}{3}t^3x^3 + t^3y^2x \Big|_0^1$$

$$= \frac{1}{3}x^3 + y^2x$$

ii) $f: \mathbb{R}^3 \to \mathbb{R}^3$ vermöge $f(x,y,z) := (x+z, -y-z, x-y)$

Aus $\frac{\partial f_1}{\partial y} = 0 = \frac{\partial f_2}{\partial x}$, $\frac{\partial f_1}{\partial z} = 1 = \frac{\partial f_3}{\partial x}$, $\frac{\partial f_2}{\partial z} = -1 = \frac{\partial f_3}{\partial y}$

& sternförmig folgt f ist Gradientenfeld

$$\varphi(x,y,z) = \int_0^1 \langle f((x_0,y_0,z_0) + t\cdot(x-x_0, y-y_0, z-z_0)), (x-x_0, y-y_0, z-z_0) \rangle \, dt$$

$$\overset{x_0,y_0,z_0=0}{=} \int_0^1 \langle f(t(x,y,z)), (x,y,z) \rangle \, dt$$

$$= \int_0^1 \langle (tx+tz, -ty-tz, tx-ty), (x,y,z) \rangle \, dt$$

$$= \int_0^1 tx^2 + tzx - ty^2 - tzy + txz - tyz \, dt$$

$$= \int_0^1 tx^2 - ty^2 + 2tzx - 2tzy \, dt$$

$$= \frac{1}{2}t^2x^2 - \frac{1}{2}t^2y^2 + t^2zx - t^2zy \Big|_0^1 = \frac{1}{2}x^2 - \frac{1}{2}y^2 + zx - zy$$