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19.4

Aufgabe 18:

$X : \Sigma \to \mathbb{R}^3$ Immersion

z.z. $|X_u(u,v) \times X_v(u,v)| = \sqrt{\det(I(x))}$

Beweis:

- $X_u = \begin{pmatrix} x_{1,u} \\ x_{2,u} \\ x_{3,u} \end{pmatrix}$, $X_v = \begin{pmatrix} x_{1,v} \\ x_{2,v} \\ x_{3,v} \end{pmatrix}$

- $Z := X_u \times X_v = \begin{pmatrix} x_{2,u} \cdot x_{3,v} - x_{3,u} \cdot x_{2,v} \\ x_{3,u} \cdot x_{1,v} - x_{1,u} \cdot x_{3,v} \\ x_{1,u} \cdot x_{2,v} - x_{2,u} \cdot x_{1,v} \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$

- $|X_u \times X_v| = |Z| = \sqrt{z_1^2 + z_2^2 + z_3^2}$

- $I(x) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$, wobei $g_{ij} = \langle X_{u^i}(u,v), X_{u^j}(u,v) \rangle$ $\quad u^1 = u$, $u^2 = v$

  und $g_{21} = g_{12}$ (wegen Eigenschaften des Skalarprodukts)

- $g_{12} = \langle X_u(u,v), X_v(u,v) \rangle$

  $= \left\langle \begin{pmatrix} x_{1,u} \\ x_{2,u} \\ x_{3,u} \end{pmatrix}, \begin{pmatrix} x_{1,v} \\ x_{2,v} \\ x_{3,v} \end{pmatrix} \right\rangle = x_{1,u} \cdot x_{1,v} + x_{2,u} \cdot x_{2,v} + x_{3,u} x_{3,v}$

- $g_{12} = g_{21}$

- $g_{11} = \langle X_u(u,v), X_u(u,v) \rangle$

  $= x_{1,u}^2 + x_{2,u}^2 + x_{3,u}^2$

- $g_{22} = \langle X_v(u,v), X_v(u,v) \rangle$

  $= x_{1,v}^2 + x_{2,v}^2 + x_{3,v}^2$

- $\sqrt{\det(I(x))}$: $\quad \det(I(x)) = \det\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = g_{11} g_{22} - g_{12} \cdot g_{21}$ $\quad // g_{12} = g_{21}$

  $= g_{11} g_{22} - g_{12}^2$

  $\rightsquigarrow \det(I(x)) = (x_{1,u}^2 + x_{2,u}^2 + x_{3,u}^2)(x_{1,v}^2 + x_{2,v}^2 + x_{3,v}^2)$

  $- (x_{1,u} \cdot x_{1,v} + x_{2,u} x_{2,v} + x_{3,u} x_{3,v})^2$

Jetzt zur eigentlichen Rechnung:

z.z. $\sqrt{\det(I(x))} = |x_u \times x_v|$

Beweis: $\det(I(x)) = |x_u \times x_v|^2 \quad \Rightarrow \quad \sqrt{\det(I(x))} = |x_u \times x_v|$

Also zeigen wir das

- $\det(I(x)) = (x_{1,u}^2 + x_{2,u}^2 + x_{3,u}^2)(x_{1,v}^2 + x_{2,v}^2 + x_{3,v}^2) - (x_{1,u}x_{1,v} + x_{2,u}x_{2,v} + x_{3,u}x_{3,v})^2$

$$|x_u \times x_v|^2 := \left|\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}\right|^2 = z_1^2 + z_2^2 + z_3^2 \overset{!}{=} \det(I(x))$$

$$\begin{aligned}
\Leftrightarrow \quad z_1^2 + z_2^2 + z_3^2 &= (x_{2,u}x_{3,v} - x_{3,u}x_{2,v})^2 \\
&\quad + (x_{3,u}x_{1,v} - x_{1,u}x_{3,v})^2 \\
&\quad + (x_{1,u}x_{2,v} - x_{2,u}x_{1,v})^2 \\
&= x_{2,u}^2\,{\scriptstyle (26)}\,x_{3,v}^2 + x_{3,u}^2\,{\scriptstyle (35)}\,x_{2,v}^2 - 2x_{2,u}x_{3,u}x_{2,v}x_{3,v} \\
&\quad + x_{3,u}^2\,{\scriptstyle (34)}\,x_{1,v}^2 + x_{1,u}^2\,{\scriptstyle (16)}\,x_{3,v}^2 - 2x_{1,u}x_{3,u}x_{1,v}x_{3,v} \\
&\quad + x_{1,u}^2\,{\scriptstyle (15)}\,x_{2,v}^2 + x_{2,u}^2\,{\scriptstyle (24)}\,x_{1,v}^2 - 2x_{1,u}x_{2,u}x_{1,v}x_{2,v} \\
&= \left(\overset{(1)}{x_{1,u}^2} + \overset{(2)}{x_{2,u}^2} + \overset{(3)}{x_{3,u}^2}\right)\left(\overset{(4)}{x_{1,v}^2} + \overset{(5)}{x_{2,v}^2} + \overset{(6)}{x_{3,v}^2}\right) \\
&\quad \left.\begin{aligned}
&- (x_{1,u}^2x_{1,v}^2 + x_{2,u}^2x_{2,v}^2 + x_{3,u}^2x_{3,v}^2) \\
&- 2x_{1,u}x_{2,u}x_{1,v}x_{2,v} \\
&- 2x_{1,u}x_{3,u}x_{1,v}x_{3,v} \\
&- 2x_{2,u}x_{3,u}x_{2,v}x_{3,v}
\end{aligned}\right\} = (x_{1,u}x_{1,v} + x_{2,u}x_{2,v} + x_{3,u}x_{3,v})^2 \\
&= \det(I(x))
\end{aligned}$$

das was hier zu viel addiert wird

$\Rightarrow \det(I(x)) = |x_u \times x_v|^2$

$\Rightarrow \sqrt{\det(I(x))} = |x_u \times x_v|$

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